

SOME INEQUALITIES FOR THE RATIOS OF GENERALIZED DIGAMMA FUNCTIONS

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ABSTRACT. Some inequalities for the ratios of generalized digamma functions are presented. The approach makes use of the series representations of the (q, k) -digamma and (p, q) -digamma functions.

1. INTRODUCTION AND PRELIMINARIES

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx, \quad \psi(t) = \frac{d}{dt} \ln \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0$$

In 2005, Díaz and Teruel [1] defined the (q, k) -Gamma function, $\Gamma_{q,k}(t)$ as

$$\Gamma_{q,k}(t) = \frac{(1 - q^k)_{q,k}^{\frac{t}{k}-1}}{(1 - q)^{\frac{t}{k}-1}} = \frac{(1 - q^k)_{q,k}^\infty}{(1 - q^t)_{q,k}^\infty (1 - q)^{\frac{t}{k}-1}}, \quad t > 0, k > 0, q \in (0, 1).$$

with the (q, k) -digamma function, $\psi_{q,k}(t)$ is defined as

$$\psi_{q,k}(t) = \frac{d}{dt} \ln \Gamma_{q,k}(t) = \frac{\Gamma'_{q,k}(t)}{\Gamma_{q,k}(t)}, \quad t > 0, k > 0, q \in (0, 1).$$

Also in 2012, Krasniqi and Merovci [2] gave the (p, q) -Gamma function, $\Gamma_{p,q}(t)$ as

$$\Gamma_{p,q}(t) = \frac{[p]_q^t [p]_q!}{[t]_q [t+1]_q \dots [t+p]_q}, \quad t > 0, p \in \mathbb{N}, q \in (0, 1).$$

where $[p]_q = \frac{1-q^p}{1-q}$.

Similarly, the (p, q) -digamma function, $\psi_{p,q}(t)$ is defined as

$$\psi_{p,q}(t) = \frac{d}{dt} \ln \Gamma_{p,q}(t) = \frac{\Gamma'_{p,q}(t)}{\Gamma_{p,q}(t)}, \quad t > 0, p \in \mathbb{N}, q \in (0, 1).$$

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The functions $\psi_{q,k}(t)$ and $\psi_{p,q}(t)$ as defined above exhibit the following series representations.

$$\psi_{q,k}(t) = \frac{-\ln(1-q)}{k} + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nkt}}{1-q^{nk}}, \quad t > 0. \quad (1)$$

$$\psi_{p,q}(t) = \ln[p]_q + (\ln q) \sum_{n=1}^p \frac{q^{nt}}{1-q^n}, \quad t > 0. \quad (2)$$

By taking derivatives of these functions, it can easily be established that,

$$\psi'_{q,k}(t) = (\ln q)^2 \sum_{n=1}^{\infty} \frac{nk \cdot q^{nkt}}{1-q^{nk}}, \quad t > 0 \quad (3)$$

$$\psi'_{p,q}(t) = (\ln q)^2 \sum_{n=1}^p \frac{n \cdot q^{nt}}{1-q^n}, \quad t > 0. \quad (4)$$

In [3], Nantomah presented the following results for the digamma function.

$$\frac{[\psi(a)]^\alpha}{[\psi(c)]^\beta} \leq \frac{[\psi(a+bt)]^\alpha}{[\psi(c+dt)]^\beta} \leq \frac{[\psi(a+b)]^\alpha}{[\psi(c+d)]^\beta} \quad (5)$$

where $a, b, c, d, \alpha, \beta$ are positive real numbers such that $\beta d \leq \alpha b$, $a+bt \leq c+dt$, $\psi(a+bt) > 0$ and $\psi(c+dt) > 0$.

Also, the k -analogue of these inequalities can be found in [4].

The purpose of this paper is to extend inequalities (5) to the (q, k) and (p, q) -digamma functions.

2. RESULTS AND DISCUSSION

We now present the results of this paper.

Lemma 2.1. *Let $0 < s \leq t$, then the following statement is valid.*

$$\psi_{q,k}(s) \leq \psi_{q,k}(t).$$

Proof. From (1) we have,

$$\psi_{q,k}(s) - \psi_{q,k}(t) = (\ln q) \sum_{n=1}^{\infty} \left[\frac{q^{nks} - q^{nkt}}{1 - q^{nk}} \right] \leq 0.$$

Lemma 2.2. *Let $0 < s \leq t$, then the following statement is valid.*

$$\psi'_{q,k}(s) \geq \psi'_{q,k}(t).$$

Proof. From (3) we have,

$$\psi'_{q,k}(s) - \psi'_{q,k}(t) = (\ln q)^2 \sum_{n=1}^{\infty} \left[\frac{nk(q^{nks} - q^{nkt})}{1 - q^{nk}} \right] \geq 0.$$

Lemma 2.3. *Let $a, b, c, d, \alpha, \beta$ be positive real numbers such that $a+bt \leq c+dt$, $\beta d \leq \alpha b$, $\psi_{q,k}(a+bt) > 0$ and $\psi_{q,k}(c+dt) > 0$. Then*

$$\alpha b \psi_{q,k}(c+dt) \psi'_{q,k}(a+bt) - \beta d \psi_{q,k}(a+bt) \psi'_{q,k}(c+dt) \geq 0.$$

Proof. Since $0 < a+bt \leq c+dt$, then by Lemmas 2.1 and 2.2 we have, $0 < \psi_{q,k}(a+bt) \leq \psi_{q,k}(c+dt)$ and $\psi'_{q,k}(a+bt) \geq \psi'_{q,k}(c+dt) > 0$.

Then that implies;

$$\psi_{q,k}(c+dt) \psi'_{q,k}(a+bt) \geq \psi_{q,k}(c+dt) \psi'_{q,k}(c+dt) \geq \psi_{q,k}(a+bt) \psi'_{q,k}(c+dt).$$

Further, $\alpha b \geq \beta d$ implies;

$$\alpha b \psi_{q,k}(c+dt) \psi'_{q,k}(a+bt) \geq \alpha b \psi_{q,k}(a+bt) \psi'_{q,k}(c+dt) \geq \beta d \psi_{q,k}(a+bt) \psi'_{q,k}(c+dt).$$

Hence,

$$\alpha b \psi_{q,k}(c+dt) \psi'_{q,k}(a+bt) - \beta d \psi_{q,k}(a+bt) \psi'_{q,k}(c+dt) \geq 0.$$

Theorem 2.4. *Define a function G by*

$$G(t) = \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta}, \quad t \in [0, \infty) \quad (6)$$

where $a, b, c, d, \alpha, \beta$ are positive real numbers such that $a+bt \leq c+dt$, $\beta d \leq \alpha b$, $\psi_{q,k}(a+bt) > 0$ and $\psi_{q,k}(c+dt) > 0$. Then G is nondecreasing on $t \in [0, \infty)$ and the inequalities

$$\frac{[\psi_{q,k}(a)]^\alpha}{[\psi_{q,k}(c)]^\beta} \leq \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta} \leq \frac{[\psi_{q,k}(a+b)]^\alpha}{[\psi_{q,k}(c+d)]^\beta} \quad (7)$$

are valid for every $t \in [0, 1]$.

Proof. Let $g(t) = \ln G(t)$ for every $t \in [0, \infty)$. Then,

$$g = \ln \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta} = \alpha \ln \psi_{q,k}(a+bt) - \beta \ln \psi_{q,k}(c+dt)$$

and

$$\begin{aligned} g'(t) &= \alpha b \frac{\psi'_{q,k}(a+bt)}{\psi_{q,k}(a+bt)} - \beta d \frac{\psi'_{q,k}(c+dt)}{\psi_{q,k}(c+dt)} \\ &= \frac{\alpha b \psi'_{q,k}(a+bt) \psi_{q,k}(c+dt) - \beta d \psi'_{q,k}(c+dt) \psi_{q,k}(a+bt)}{\psi_{q,k}(a+bt) \psi_{q,k}(c+dt)} \geq 0 \end{aligned}$$

as a result of Lemma 2.3. That implies g as well as G are nondecreasing on $t \in [0, \infty)$ and for every $t \in [0, 1]$ we have,

$$G(0) \leq G(t) \leq G(1)$$

concluding the proof.

Corollary 2.5. *If $t \in (1, \infty)$, then the following inequality is valid.*

$$\frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta} \geq \frac{[\psi_{q,k}(a+b)]^\alpha}{[\psi_{q,k}(c+d)]^\beta} \quad (8)$$

Proof. For each $t \in (1, \infty)$, we have $G(t) \geq G(1)$ yielding the result.

Lemma 2.6. *Let $0 < s \leq t$, then the following statement is valid.*

$$\psi_{p,q}(s) \leq \psi_{p,q}(t).$$

Proof. From (2) we have,

$$\psi_{p,q}(s) - \psi_{p,q}(t) = (\ln q) \sum_{n=1}^p \left[\frac{q^{ns} - q^{nt}}{1 - q^n} \right] \leq 0.$$

Lemma 2.7. *Let $0 < s \leq t$, then the following statement is valid.*

$$\psi'_{p,q}(s) \geq \psi'_{p,q}(t).$$

Proof. From (4) we have,

$$\psi'_{p,q}(s) - \psi'_{p,q}(t) = (\ln q)^2 \sum_{n=1}^p \left[\frac{n(q^{ns} - q^{nt})}{1 - q^n} \right] \geq 0.$$

Lemma 2.8. *Let $a, b, c, d, \alpha, \beta$ be positive real numbers such that $a + bt \leq c + dt$, $\beta d \leq \alpha b$, $\psi_{p,q}(a + bt) > 0$ and $\psi_{p,q}(c + dt) > 0$. Then*

$$\alpha b \psi_{p,q}(c + dt) \psi'_{p,q}(a + bt) - \beta d \psi_{p,q}(a + bt) \psi'_{p,q}(c + dt) \geq 0.$$

Proof. Follows the same argument as in the proof of Lemma 2.3.

Theorem 2.9. *Define a function H by*

$$H(t) = \frac{[\psi_{p,q}(a + bt)]^\alpha}{[\psi_{p,q}(c + dt)]^\beta}, \quad t \in [0, \infty) \quad (9)$$

where $a, b, c, d, \alpha, \beta$ are positive real numbers such that $a + bt \leq c + dt$, $\beta d \leq \alpha b$, $\psi_{p,q}(a + bt) > 0$ and $\psi_{p,q}(c + dt) > 0$. Then H is nondecreasing on $t \in [0, \infty)$ and the inequalities

$$\frac{[\psi_{p,q}(a)]^\alpha}{[\psi_{p,q}(c)]^\beta} \leq \frac{[\psi_{p,q}(a + bt)]^\alpha}{[\psi_{p,q}(c + dt)]^\beta} \leq \frac{[\psi_{p,q}(a + b)]^\alpha}{[\psi_{p,q}(c + d)]^\beta} \quad (10)$$

are valid for every $t \in [0, 1]$.

Proof. Follows the same procedure as in Theorem 2.4. Using Lemma 2.8, we conclude that H is nondecreasing on $t \in [0, \infty)$ and for every $t \in [0, 1]$ we have,

$$H(0) \leq H(t) \leq H(1)$$

ending the proof.

Corollary 2.10. *If $t \in (1, \infty)$, then the following inequality is valid.*

$$\frac{[\psi_{p,q}(a + bt)]^\alpha}{[\psi_{p,q}(c + dt)]^\beta} \geq \frac{[\psi_{p,q}(a + b)]^\alpha}{[\psi_{p,q}(c + d)]^\beta} \quad (11)$$

Proof. For each $t \in (1, \infty)$, we have $H(t) \geq H(1)$ yielding the result.

3. CONCLUDING REMARKS

We dedicate this section to some remarks concerning our results.

Remark 3.1. If in (7) we allow $k \rightarrow 1$, then we obtain the q -analogue of (5).

Remark 3.2. If in (7) we allow $q \rightarrow 1^-$, then we obtain the k -analogue of (5) as presented in Theorem 3.7 of the paper [4].

Remark 3.3. If in (7) we allow $q \rightarrow 1^-$ as $k \rightarrow 1$, then we obtain (5).

Remark 3.4. If in (10) we allow $q \rightarrow 1^-$, then we obtain the p -analogue of (5).

Remark 3.5. If in (10) we allow $p \rightarrow \infty$, then we obtain the q -analogue of (5).

Remark 3.6. If in (10) we allow $p \rightarrow \infty$ as $q \rightarrow 1^-$, then we obtain (5).

Conflict of Interests. The authors declare that there is no conflict of interests.

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